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Four-wave interaction equations: Hamiltonian formulation, conservation laws and the Hirota method

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Abstract. Some of the analytical properties of the four-wave interaction equations derived previously by the present authors are studied. The equations are shown to be Hamiltonian and to have, at least, four conserved densities. Solitary wave solutions of the equations are also considered by means of a modified Hirota method.

1. Introduction

The linear dispersion relation corresponding to transverse waves in micropolar elastic solids has two branches, the so-called acoustical branch which involves low frequencies and the optical branch which involves a range of higher frequencies [1]. In [2], making no distinction between transverse acoustical and optical branches and using the reductive perturbation method, Erbay *et al* show that the slowly varying complex amplitudes of the transverse (displacement or microrotation) waves are governed by the following set of two coupled nonlinear Schrödinger (NLS2) equations:

$$\begin{aligned} &\mathrm{i}\phi_\tau + \Gamma_T \phi_{\xi\xi} + [\Delta_1|\phi|^2 + (\Delta_1 + 2\Delta_2)|\psi|^2]\phi = 0 \\ &\mathrm{i}\psi_\tau + \Gamma_T \psi_{\xi\xi} + [\Delta_1|\psi|^2 + (\Delta_1 + 2\Delta_2)|\phi|^2]\psi = 0 \end{aligned} \tag{1}$$

where the variables τ and ξ are the dimensionless time and space coordinates, respectively, in a reference frame moving at the group velocity, and the coefficients Γ_T, Δ_1 and Δ_2 are real functions of wavenumber and material parameters. Here and hereafter, subscripts τ and ξ denote partial differentiations. The functions ϕ and ψ are the complex amplitudes of two circularly-polarized transverse waves and are written in terms of the complex amplitudes, Φ and Ψ, of two linearly-polarized transverse waves as follows:

$$\phi = (\Phi - \mathrm{i}\Psi)/\sqrt{2} \qquad \psi = (\Phi + \mathrm{i}\Psi)/\sqrt{2}.$$

Note that in (1), the nonlinear interaction between the transverse waves belonging to one of the branches (acoustical or optical) has been considered only. It has been shown by Zakharov and Schulman [3] that the NLS2 equations are not in general integrable by inverse scattering transform.

In [4] the nonlinear interaction between modulated transverse acoustical and optical waves having equal group velocity has been investigated. In such a case, there exist four transverse modes simultaneously present in the medium. While each of these modes propagates at the same group velocity, acoustical and optical waves can have different

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wavenumbers and phase velocities. Using the reductive perturbation method, it is shown that the slow modulation of the complex amplitudes of circularly-polarized transverse waves is described by the following four coupled nonlinear evolution equations:

$$
\begin{aligned}
&\mathrm{i}\phi_{1\tau} + \Gamma_1\phi_{1\xi\xi} + [\Delta_{11}|\phi_1|^2 + (\Delta_{11} + 2\Delta_{12})|\psi_1|^2 + \nu_{11}|\phi_2|^2 + \nu_{12}|\psi_2|^2]\phi_1 \\
&\qquad + (\nu_{13}\phi_2\psi_2^* + \nu_{14}\phi_2^*\psi_2)\psi_1 = 0 \\
&\mathrm{i}\psi_{1\tau} + \Gamma_1\psi_{1\xi\xi} + [\Delta_{11}|\psi_1|^2 + (\Delta_{11} + 2\Delta_{12})|\phi_1|^2 + \nu_{11}|\psi_2|^2 + \nu_{12}|\phi_2|^2]\psi_1 \\
&\qquad + (\nu_{13}\phi_2^*\psi_2 + \nu_{14}\phi_2\psi_2^*)\phi_1 = 0 \\
&\mathrm{i}\phi_{2\tau} + \Gamma_2\phi_{2\xi\xi} + [\Delta_{21}|\phi_2|^2 + (\Delta_{21} + 2\Delta_{22})|\psi_2|^2 + \nu_{21}|\phi_1|^2 + \nu_{22}|\psi_1|^2]\phi_2 \\
&\qquad + (\nu_{23}\phi_1\psi_1^* + \nu_{24}\phi_1^*\psi_1)\psi_2 = 0 \\
&\mathrm{i}\psi_{2\tau} + \Gamma_2\psi_{2\xi\xi} + [\Delta_{21}|\psi_2|^2 + (\Delta_{21} + 2\Delta_{22})|\phi_2|^2 + \nu_{21}|\psi_1|^2 + \nu_{22}|\phi_1|^2]\psi_2 \\
&\qquad + (\nu_{23}\phi_1^*\psi_1 + \nu_{24}\phi_1\psi_1^*)\phi_2 = 0
\end{aligned}
\tag{2}
$$

where ϕ_j and ψ_j $(j = 1, 2)$ (here and hereafter, j takes the values 1 and 2 only and subscripts 1 and 2 correspond to acoustical and optical branches, respectively) represent the complex amplitudes of two pairs of two circularly-polarized transverse waves and the coefficients Γ_j, Δ_{j1}, Δ_{j2}, ν_{j1}, ν_{j2}, ν_{j3} and ν_{j4} $(j = 1, 2)$ are real functions of wavenumbers and material parameters. In the above four-wave interaction (FWI) equations, the second terms represent the dispersive effect and the nonlinear terms are of two types, i.e. the first two terms inside the brackets and all the remaining nonlinear terms. The first two terms inside the brackets describe the interaction of a wave with itself and the mutual interaction of pairs of waves belonging to the same branch, respectively. However, all the remaining nonlinear terms describe the interaction between waves belonging to two different branches. It should be also pointed out that, in the absence of the last terms, the FWI equations reduce to a system of four coupled nonlinear Schrödinger (NLS4) equations.

In the present study, we establish some properties of the FWI equations given by (2). We show that the FWI system is a Hamiltonian system and admits at least four conservation laws. We also apply a modified Hirota method and obtain particular exact solutions of the system for certain choices of the coefficients. These solutions reduce to hyperbolic secant type envelope solitary wave solutions at the decoupled case in which acoustical or optical waves only exist in the medium.

2. Hamiltonian formulation and conservation laws

The NLS2 equations are already known to be Hamiltonian [5]. Furthermore, for a special form of the FWI equations and without considering spatial variations, a complex Hamiltonian formalism is given in [6]. In this section, we first consider the Hamiltonian nature of the FWI equations and show that system (2) is a Hamiltonian system.

System (2) is derivable from a Lagrangian density $\mathcal{L}$ provided the coupling coefficients ν_{jl} $(j = 1, 2; l = 1, 2, 3, 4)$ satisfy the following conditions:

$$\nu_{11} = \nu_{21} \qquad \nu_{12} = \nu_{22} \qquad \nu_{13} = \nu_{23} \qquad \nu_{14} = \nu_{24}. \tag{3}$$

In such a case, $\mathcal{L}$ is given by

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 - \mathcal{L}_{12} \tag{4}$$

where $\mathcal{L}_j$ $(j = 1, 2)$ corresponds to the NLS2 Lagrangian

$$
\begin{aligned}
\mathcal{L}_j = {}& \tfrac{1}{2}\mathrm{i}(\phi_j\phi_{j\tau}^* - \phi_j^*\phi_{j\tau} + \psi_j\psi_{j\tau}^* - \psi_j^*\psi_{j\tau}) + \Gamma_j(|\phi_{j\xi}|^2 + |\psi_{j\xi}|^2) \\
& - \tfrac{1}{2}\Delta_{j1}(|\phi_j|^4 + |\psi_j|^4) - (\Delta_{j1} + 2\Delta_{j2})|\phi_j|^2|\psi_j|^2
\end{aligned}
$$

in which the last term represents the interaction between acoustical waves ($j = 1$) or optical waves ($j = 2$), and $\mathcal{L}_{12}$ represents the interaction Lagrangian

$$\mathcal{L}_{12} = \nu_{11}(|\phi_1|^2|\phi_2|^2 + |\psi_1|^2|\psi_2|^2) + \nu_{12}(|\phi_1|^2|\psi_2|^2 + |\phi_2|^2|\psi_1|^2) \\ +\nu_{13}(\phi_1^*\psi_1\phi_2\psi_2^* + \phi_1\psi_1^*\phi_2^*\psi_2) + \nu_{14}(\phi_1^*\psi_1\phi_2^*\psi_2 + \phi_1\psi_1^*\phi_2\psi_2^*)$$

which includes the terms resulting from the interaction between acoustical and optical waves only. Note that $\mathcal{L}_j$ $(j = 1, 2)$ and $\mathcal{L}_{12}$ are real functions of their complex arguments.

Canonical momenta $\pi_j(\xi, \tau)$, $\pi_j^*(\xi, \tau)$, $\Pi_j(\xi, \tau)$ and $\Pi_j^*(\xi, \tau)$ $(j = 1, 2)$ are defined by

$$\pi_j = \frac{\partial \mathcal{L}}{\partial \phi_{j\tau}} = -\frac{\mathrm{i}}{2}\phi_j^* \qquad \pi_j^* = \frac{\partial \mathcal{L}}{\partial \phi_{j\tau}^*} = \frac{\mathrm{i}}{2}\phi_j \\ \Pi_j = \frac{\partial \mathcal{L}}{\partial \psi_{j\tau}} = -\frac{\mathrm{i}}{2}\psi_j^* \qquad \Pi_j^* = \frac{\partial \mathcal{L}}{\partial \psi_{j\tau}^*} = \frac{\mathrm{i}}{2}\psi_j. \tag{5}$$

Then, the Hamiltonian density $\mathcal{H}$ and the Hamiltonian H are given by

$$\mathcal{H} = \sum_{j=1}^{2}(\pi_j\phi_{j\tau} + \Pi_j\psi_{j\tau} + \pi_j^*\phi_{j\tau}^* + \Pi_j^*\psi_{j\tau}^*) - \mathcal{L} \\ = \mathcal{L}_{12} + \sum_{j=1}^{2}[-\Gamma_j(|\phi_{j\xi}|^2 + |\psi_{j\xi}|^2) + \tfrac{1}{2}\Delta_{j1}(|\phi_j|^4 + |\psi_j|^4) \\ +(\Delta_{j1} + 2\Delta_{j2})|\phi_j|^2|\psi_j|^2] \tag{6}$$

and

$$H = \int \mathcal{H}\,\mathrm{d}\xi$$

respectively. From the Hamiltonian formulation of the variational problem, the following Hamilton canonical equations are obtained

$$\phi_{j\tau} = \mathrm{i}\frac{\delta H}{\delta\phi_j^*} \qquad \psi_{j\tau} = \mathrm{i}\frac{\delta H}{\delta\psi_j^*} \qquad \phi_{j\tau}^* = -\mathrm{i}\frac{\delta H}{\delta\phi_j} \qquad \psi_{j\tau}^* = -\mathrm{i}\frac{\delta H}{\delta\psi_j} \qquad (j = 1, 2) \tag{7}$$

which implies that the complex wave amplitudes and their complex conjugates serve as the canonical variables. The Poisson bracket is defined in terms of these conjugate variables (ϕ_j, ϕ_j^*) and (ψ_j, ψ_j^*) as follows:

$$\{F, G\} = \mathrm{i}\int \sum_{j=1}^{2}\left(\frac{\delta F}{\delta\phi_j}\frac{\delta G}{\delta\phi_j^*} - \frac{\delta F}{\delta\phi_j^*}\frac{\delta G}{\delta\phi_j} + \frac{\delta F}{\delta\psi_j}\frac{\delta G}{\delta\psi_j^*} - \frac{\delta F}{\delta\psi_j^*}\frac{\delta G}{\delta\psi_j}\right)\mathrm{d}\xi. \tag{8}$$

It is possible to prove by direct calculations that system (2) is recovered from the Hamiltonian formalism. From (6)–(8), we have

$$\phi_{j\tau} = \{\phi_j, H\} = \mathrm{i}\frac{\delta H}{\delta\phi_j^*} \qquad \psi_{j\tau} = \{\psi_j, H\} = \mathrm{i}\frac{\delta H}{\delta\psi_j^*} \qquad (j = 1, 2) \\ \phi_{j\tau}^* = \{\phi_j^*, H\} = -\mathrm{i}\frac{\delta H}{\delta\phi_j} \qquad \psi_{j\tau}^* = \{\psi_j^*, H\} = -\mathrm{i}\frac{\delta H}{\delta\psi_j} \qquad (j = 1, 2). \tag{9}$$

These equations yield system (2) and its complex conjugate. This confirms that system (2) is a Hamiltonian system.

We may ask whether system (2) is integrable or not. It is well known that there exist an infinite number of conservation laws for most integrable Hamiltonian nonlinear partial differential equations. However, only a few of these conservation laws seem to bear any plausible physical interpretation. Even if the number of conservation laws is limited, the information may be useful for analytic and numerical computations.

The construction of conserved quantities is based on symmetry arguments. In other words, the symmetries of system (2) are of fundamental importance to the analysis. System (2) is clearly translation invariant in ξ and τ. Morover, it has the following gauge symmetry; that is $(e^{i\alpha}\phi_1, e^{i\alpha}\psi_1, e^{i\beta}\phi_2, e^{i\beta}\psi_2)$ is a solution of system (2) whenever $(\phi_1, \psi_1, \phi_2, \psi_2)$ is a solution. The four continuous symmetries (translation in τ, translation in ξ and the gauge symmetry) generate, by Noether's theorem, four conservation laws. These conservation laws are given as follows:

$$\frac{\partial D_k}{\partial \tau} + \frac{\partial F_k}{\partial \xi} = 0 \qquad (k = 1, 2, 3, 4) \tag{10}$$

in which the conserved densities, D_k, and the fluxes, F_k, take the form

$$\begin{aligned}
D_j &= |\phi_j|^2 + |\psi_j|^2 \qquad (j = 1, 2)\\
D_3 &= i\sum_{j=1}^{2}(\phi_j^*\phi_{j\xi} - \phi_j\phi_{j\xi}^* + \psi_j^*\psi_{j\xi} - \psi_j\psi_{j\xi}^*)\\
D_4 &= \mathcal{L}_{12} + \sum_{j=1}^{2}[-\Gamma_j(|\phi_{j\xi}|^2 + |\psi_{j\xi}|^2) + \tfrac{1}{2}\Delta_{j1}(|\phi_j|^4 + |\psi_j|^4) + (\Delta_{j1} + 2\Delta_{j2})|\phi_j|^2|\psi_j|^2]
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
F_j &= i\Gamma_j(\phi_j\phi_{j\xi}^* - \phi_j^*\phi_{j\xi} + \psi_j\psi_{j\xi}^* - \psi_j^*\psi_{j\xi}) \qquad (j = 1, 2)\\
F_3 &= 2\mathcal{L}_{12} + \sum_{j=1}^{2}\{\Gamma_j[\phi_j^*\phi_{j\xi\xi} + \phi_j\phi_{j\xi\xi}^* + \psi_j^*\psi_{j\xi\xi} + \psi_j\psi_{j\xi\xi}^* - 2(|\phi_{j\xi}|^2 + |\psi_{j\xi}|^2)]\\
&\qquad + \Delta_{j1}(|\phi_j|^4 + |\psi_j|^4) + 2(\Delta_{j1} + 2\Delta_{j2})|\phi_j|^2|\psi_j|^2\}\\
F_4 &= \sum_{j=1}^{2}\Gamma_j(\phi_{j\tau}^*\phi_{j\xi} + \phi_{j\tau}\phi_{j\xi}^* + \psi_{j\tau}^*\psi_{j\xi} + \psi_{j\tau}\psi_{j\xi}^*)
\end{aligned} \tag{12}$$

respectively. By carrying out the operations in (11) and (12) and using system (2) it can be verified that (11) and (12) are formally conserved densities and fluxes of the conservation laws for system (2). D_1 and D_2 are the total densities of acoustical and optical waves, respectively, D_3 is the total momentum density and D_4 is the total energy density of the system. The first two conservation laws are associated with the gauge symmetry. Whereas the momentum conservation law is associated with the translation invariance of the system in space, the energy conservation law is associated with the translation invariance of the system in time. We cannot conclude whether there exist further conservation laws.

Finally we consider the special case where $\nu_{j3} = \nu_{j4} = 0$ $(j = 1, 2)$ in which, as we have remarked already, the FWI system reduces to the NLS4 equations. In such a case, as shown in [4], each wave density is conserved, i.e. the NLS4 equations have at least six conserved quantities, $\overline{D}_j$, instead of four: $\overline{D}_j = |\phi_j|^2$ $(j = 1, 2)$, $\overline{D}_j = |\psi_j|^2$ $(j = 3, 4)$, $\overline{D}_5 = D_3$ and $\overline{D}_6 = D_4$. The reason for this situation is that the NLS4 equations have six continuous symmetries. These are translation in τ, translation in ξ and the following gauge symmetry; that is $(e^{i\alpha_1}\phi_1, e^{i\beta_1}\psi_1, e^{i\alpha_2}\phi_2, e^{i\beta_2}\psi_2)$ is a solution of the NLS4 equations whenever $(\phi_1, \psi_1, \phi_2, \psi_2)$ is a solution.

3. The modified Hirota method

In [7] and [8] a modified Hirota method has been applied to the NLS2 equations with a linear birefringence term in order to obtain a new family of solitary wave solutions. This new family of solitary waves reduces to the classical solitary wave solution of the single nonlinear Schrödinger (NLS1) equation when one of the complex amplitudes vanishes or two complex amplitudes are equal to each other. In other words, the NLS2 system admits particular exact solutions, i.e. the mixed-type solutions, which are the well known solitary wave solutions of the NLS1 equation at the decoupled case. It should also be pointed out that the linear birefringence term added to the NLS2 equations in [7] and [8] is crucial for the existence of a qualitatively new family of solitary waves which in its absence reduce to the single soliton family of the NLS1 equation. The unusual characteristic about the modified Hirota method in comparison to the classical one [9] is that it begins with a reduction of the original partial differential equations to the ordinary differential equations by means of a travelling-wave transformation. Then it involves a series expansion of dependent variables. As in the classical one, one of the most important aspects of the modified Hirota method is that it may be possible to truncate the series expansion at a finite number of terms. In this section we study special solutions of system (2) by using the modified Hirota method developed in [7] and [8].

We begin by introducing the following simple travelling-wave transformation

$$\phi_j(\xi,\tau) = \bar{\phi}_j(\zeta)\exp(\mathrm{i}\chi_j) \qquad \psi_j(\xi,\tau) = \bar{\psi}_j(\zeta)\exp(\mathrm{i}\chi_j)$$
$$\chi_j(\xi,\tau) = \Gamma_j C_j^2\tau + \frac{v_0}{2\Gamma_j}\left(\xi - \frac{v_0}{2}\tau\right) \qquad (j=1,2) \tag{13}$$

where v_0, C_1 and C_2 are real constants and $\bar{\phi}_j$ and $\bar{\psi}_j$ $(j = 1, 2)$ are real functions of $\zeta = \xi - v_0\tau$ alone. Inspection of exactly known limiting cases guides the selection of a suitable ansatz. The simplest limiting case is the one in which there is no interaction between the modes and system (2) reduces to four uncoupled NLS1 equations. Thus each field in the absence of the others may accept the envelope solitary wave solution, with a hyperbolic secant type intensity, of the NLS1 equation. It should also be noted that acoustical $(j = 1)$ and optical $(j = 2)$ waves have different phases (i.e. $C_1 \neq C_2$ and $\Gamma_1 \neq \Gamma_2$), whereas the two fields corresponding to each branch are phase locked, which is dictated by the last terms of system (2). The next step is to substitute the ansatz (13) into system (2) and use Hirota's approach to solve the resulting ordinary differential equations. However, this works for some special values of the coefficients in system (2). For the sake of convenience we assume that the restrictions imposed on the coefficients of system (2) by the method are satisfied and we consider the following form of system (2):

$$\mathrm{i}\phi_{1\tau} + \Gamma_1\phi_{1\xi\xi} + \Gamma_1 V(|\phi_j|^2, |\psi_j|^2)\phi_1 + \gamma_1(\phi_2\psi_2^* - \phi_2^*\psi_2)\psi_1 = 0$$
$$\mathrm{i}\psi_{1\tau} + \Gamma_1\psi_{1\xi\xi} + \Gamma_1 V(|\phi_j|^2, |\psi_j|^2)\psi_1 + \gamma_1(\phi_2^*\psi_2 - \phi_2\psi_2^*)\phi_1 = 0$$
$$\mathrm{i}\phi_{2\tau} + \Gamma_2\phi_{2\xi\xi} + \Gamma_2 V(|\phi_j|^2, |\psi_j|^2)\phi_2 + \gamma_2(\phi_1\psi_1^* - \phi_1^*\psi_1)\psi_2 = 0$$
$$\mathrm{i}\psi_{2\tau} + \Gamma_2\psi_{2\xi\xi} + \Gamma_2 V(|\phi_j|^2, |\psi_j|^2)\psi_2 + \gamma_2(\phi_1^*\psi_1 - \phi_1\psi_1^*)\phi_2 = 0 \tag{14}$$

where the function $V(|\phi_j|^2, |\psi_j|^2)$ is defined by

$$V(|\phi_j|^2, |\psi_j|^2) = \delta_1(|\phi_1|^2 + |\psi_1|^2) + \delta_2(|\phi_2|^2 + |\psi_2|^2).$$

Note that in order to reduce system (2) to system (14) the following relations must be satisfied:

$$\delta_1 = \frac{\Delta_{11}}{\Gamma_1} = \frac{\nu_{21}}{\Gamma_2} = \frac{\nu_{22}}{\Gamma_2} \qquad \delta_2 = \frac{\Delta_{21}}{\Gamma_2} = \frac{\nu_{11}}{\Gamma_1} = \frac{\nu_{12}}{\Gamma_1}$$

$$\gamma_1 = \nu_{13} = -\nu_{14} \qquad \gamma_2 = \nu_{23} = -\nu_{24} \qquad \Delta_{12} = \Delta_{22} = 0.$$

Now substituting the ansatz (13) into system (14) we obtain

$$\begin{aligned} &\overline{\phi}_{j\zeta\zeta} - C_j^2\overline{\phi}_j + V(\overline{\phi}_j, \overline{\Psi}_j)\overline{\phi}_j = 0 \qquad (j = 1, 2) \\ &\overline{\Psi}_{j\zeta\zeta} - C_j^2\overline{\Psi}_j + V(\overline{\phi}_j, \overline{\Psi}_j)\overline{\Psi}_j = 0 \qquad (j = 1, 2) \end{aligned} \tag{15}$$

where the function V is

$$V(\overline{\phi}_j, \overline{\Psi}_j) = \delta_1(\overline{\phi}_1^2 + \overline{\Psi}_1^2) + \delta_2(\overline{\phi}_2^2 + \overline{\Psi}_2^2).$$

To solve system (15) we apply the modified method by assuming the following forms of $\overline{\phi}_j$ and $\overline{\Psi}_j$ in terms of real functions $f(\zeta)$, $g_j(\zeta)$ and $h_j(\zeta)$ $(j = 1, 2)$:

$$\overline{\phi}_j(\zeta) = \frac{g_j(\zeta)}{f(\zeta)} \qquad \overline{\Psi}_j(\zeta) = \frac{h_j(\zeta)}{f(\zeta)} \qquad (j = 1, 2) \tag{16}$$

where $f(\zeta)$ satisfies the relation

$$V(\overline{\phi}_j, \overline{\Psi}_j) = 2(\ln f)_{\zeta\zeta}. \tag{17}$$

Using (15)–(17) we obtain a set of five coupled equations for f, g_j and h_j, namely

$$\begin{aligned} &f g_{j\zeta\zeta} + g_j f_{\zeta\zeta} - 2 f_\zeta g_{j\zeta} = C_j^2 f g_j \qquad (j = 1, 2) \\ &f h_{j\zeta\zeta} + h_j f_{\zeta\zeta} - 2 f_\zeta h_{j\zeta} = C_j^2 f h_j \qquad (j = 1, 2) \\ &\delta_1(g_1^2 + h_1^2) + \delta_2(g_2^2 + h_2^2) = 2 f^2(\ln f)_{\zeta\zeta}. \end{aligned} \tag{18}$$

We look for solutions to these equations in the form of a power series in a parameter ϵ which is introduced simply to keep track of the terms in the expansion:

$$f = 1 + \sum_{m=1}^{\infty} f^{(n)}\epsilon^n \qquad g_j = \sum_{m=1}^{\infty} g_j{}^{(n)}\epsilon^n \qquad h_j = \sum_{m=1}^{\infty} h_j{}^{(n)}\epsilon^n. \tag{19}$$

Solutions such that the series terminates after a finite number of terms will now be looked for. Substitution of (19) into system (18) gives the following first-order problem

$$f_{\zeta\zeta}^{(1)} = 0 \qquad g_{j\zeta\zeta}{}^{(1)} - C_j^2 g_j{}^{(1)} = 0 \qquad h_{j\zeta\zeta}{}^{(1)} - C_j^2 h_j{}^{(1)} = 0$$

for which a solution is

$$f^{(1)} = 0 \qquad g_j{}^{(1)} = 2A_j \exp(\theta_j) \qquad h_j{}^{(1)} = 2B_j \exp(\theta_j) \tag{20}$$

where

$$\theta_1 = C_1(\zeta - \zeta_1) \qquad \theta_2 = C_2(\zeta - \zeta_2)$$

with ζ_1 and ζ_2 arbitrary constants.

For second order in ϵ, taking into account (20), we find

$$f_{\zeta\zeta}{}^{(2)} = 2\delta_1(A_1^2 + B_1^2)\exp(2\theta_1) + 2\delta_2(A_2^2 + B_2^2)\exp(2\theta_2)$$

$$g_{j\zeta\zeta}{}^{(2)} - C_j^2 g_j{}^{(2)} = 0 \qquad h_{j\zeta\zeta}{}^{(2)} - C_j^2 h_j{}^{(2)} = 0$$

which gives

$$f^{(2)} = \sigma_1 \exp(2\theta_1) + \sigma_2 \exp(2\theta_2) \qquad g_j{}^{(2)} = 0 \qquad h_j{}^{(2)} = 0 \tag{21}$$

where

$$\sigma_j = \frac{\delta_j(A_j^2 + B_j^2)}{2C_j^2}.$$

We now proceed to the third-order problem. Using the results (20) and (21) previously obtained we find

$$f_{\zeta\zeta}^{(3)} = 0$$
$$g_{1\zeta\zeta}{}^{(3)} - C_1^2 g_1{}^{(3)} = 8A_1\sigma_2 C_2(C_1 - C_2)\exp(\theta_1 + 2\theta_2)$$
$$h_{1\zeta\zeta}{}^{(3)} - C_1^2 h_1{}^{(3)} = 8B_1\sigma_2 C_2(C_1 - C_2)\exp(\theta_1 + 2\theta_2)$$
$$g_{2\zeta\zeta}{}^{(3)} - C_2^2 g_2{}^{(3)} = -8A_2\sigma_1 C_1(C_1 - C_2)\exp(2\theta_1 + \theta_2)$$
$$h_{2\zeta\zeta}{}^{(3)} - C_2^2 h_2{}^{(3)} = -8B_2\sigma_1 C_1(C_1 - C_2)\exp(2\theta_1 + \theta_2)$$

which gives

$$f^{(3)} = 0$$
$$(g_1{}^{(3)}, h_1{}^{(3)}) = 2(A_1, B_1)\sigma_2 \frac{C_1 - C_2}{C_1 + C_2}\exp(\theta_1 + 2\theta_2)$$
$$(g_2{}^{(3)}, h_2{}^{(3)}) = -2(A_2, B_2)\sigma_1 \frac{C_1 - C_2}{C_1 + C_2}\exp(2\theta_1 + \theta_2).$$

In a similar manner we get the following fourth-order problem

$$f_{\zeta\zeta}^{(4)} = 4\sigma_1\sigma_2(C_1 - C_2)^2\exp(2\theta_1 + 2\theta_2)$$
$$g_{j\zeta\zeta}{}^{(4)} - C_j^2 g_j{}^{(4)} = 0 \qquad h_{j\zeta\zeta}{}^{(4)} - C_j^2 h_j{}^{(4)} = 0$$

which gives

$$f^{(4)} = \sigma_1\sigma_2\left(\frac{C_1 - C_2}{C_1 + C_2}\right)^2\exp(2\theta_1 + 2\theta_2) \qquad g_j{}^{(4)} = 0 \qquad h_j{}^{(4)} = 0.$$

Simple but tedious algebra then shows that the right-hand side of the fifth-order problem is identically zero so that we can choose $f^{(5)} = g_j{}^{(5)} = h_j{}^{(5)} = 0$. Furthermore, at this point we assume that the series can be truncated, that is all the higher order terms can be set to zero. Then putting $\epsilon = 1$ we have an exact solution of (18) in the form

$$f(\zeta) = 1 + \sigma_1\exp(2\theta_1) + \sigma_2\exp(2\theta_2) + \sigma_1\sigma_2\left(\frac{C_1 - C_2}{C_1 + C_2}\right)^2\exp(2\theta_1 + 2\theta_2)$$
$$g_j(\zeta) = A_j v_j(\zeta) \qquad h_j(\zeta) = B_j v_j(\zeta) \tag{22}$$

where

$$v_1(\zeta) = 2\exp(\theta_1)\left[1 + \sigma_2\frac{C_1 - C_2}{C_1 + C_2}\exp(2\theta_2)\right]$$
$$v_2(\zeta) = 2\exp(\theta_2)\left[1 - \sigma_1\frac{C_1 - C_2}{C_1 + C_2}\exp(2\theta_1)\right].$$

Substitution of (22) into (18) verifies that this is indeed a solution, thus justifying our assumption.

If the following conditions are assumed:

$$C_j^2 = \frac{\delta_j}{2}(A_j^2 + B_j^2) \tag{23}$$

in which $\sigma_1 = \sigma_2 = 1$, we obtain from (22) and (16)

$$\overline{\phi}_j(\zeta) = A_j u_j(\zeta) \qquad \overline{\Psi}_j(\zeta) = B_j u_j(\zeta) \tag{24}$$

or, from (13), explicitly

$$\phi_j(\xi,\tau) = A_j u_j(\xi,\tau) \exp\left\{ \mathrm{i}\left[\Gamma_j C_j^2 \tau + \frac{v_0}{2\Gamma_j}(\xi - \frac{v_0}{2}\tau) \right] \right\}$$
$$\psi_j(\xi,\tau) = B_j u_j(\xi,\tau) \exp\left\{ \mathrm{i}\left[\Gamma_j C_j^2 \tau + \frac{v_0}{2\Gamma_j}(\xi - \frac{v_0}{2}\tau) \right] \right\} \tag{25}$$

where u_1 and u_2 are given by

$$u_1(\zeta) = \frac{2\exp(\theta_1)[1 + ((C_1 - C_2)/(C_1 + C_2))\exp(2\theta_2)]}{1 + \exp(2\theta_1) + \exp(2\theta_2) + ((C_1 - C_2)/(C_1 + C_2))^2 \exp(2\theta_1 + 2\theta_2)}$$
$$u_2(\zeta) = \frac{2\exp(\theta_2)[1 - ((C_1 - C_2)/(C_1 + C_2))\exp(2\theta_1)]}{1 + \exp(2\theta_1) + \exp(2\theta_2) + ((C_1 - C_2)/(C_1 + C_2))^2 \exp(2\theta_1 + 2\theta_2)}. \tag{26}$$

These solutions are very similar to those obtained for the NLS2 equations with a linear birefringence term in [7] and [8]. However, while they show the mixed-type solutions, which are bound states of two solitary waves which separately have constant and uniform orthogonal linear polarizations, in [7] and [8], here they show the mixed-type solutions which are bound states of two solitary waves which separately belong to two different (i.e. acoustical and optical) branches. In order to see this we consider two special cases in which system (14) reduces to the NLS2 equations.

Case (i). There exist waves corresponding to the acoustical branch only in the medium. That is, $A_2 = B_2 = 0$ and consequently $\bar{\phi}_2(\zeta) = \bar{\psi}_2(\zeta) = 0$. In such a case, $C_2 = 0$ and we obtain the following solitary wave solution:

$$\bar{\phi}_1(\zeta) = A_1 \operatorname{sech}[C_1(\zeta - \zeta_1)] \qquad \bar{\psi}_1(\zeta) = B_1 \operatorname{sech}[C_1(\zeta - \zeta_1)] \tag{27}$$

where C_1 is given by (23). Thus (27) describes the solitary wave solution with position of maximum ζ_1 and different A_1 and B_1 amplitudes in the transverse directions.

Case (ii). There exist optical waves in the medium only. That is, $A_1 = B_1 = 0$ and consequently $\bar{\phi}_1(\zeta) = \bar{\psi}_1(\zeta) = 0$. In such a case, $C_1 = 0$ and the following solitary wave solution is obtained:

$$\bar{\phi}_2(\zeta) = A_2 \operatorname{sech}[C_2(\zeta - \zeta_2)] \qquad \bar{\psi}_2(\zeta) = B_2 \operatorname{sech}[C_2(\zeta - \zeta_2)] \tag{28}$$

where C_2 is given by (23). Thus (28) represents the solitary wave solution with position of maximum ζ_2 and different A_2 and B_2 amplitudes in the transverse directions.

Then one can think that (25) describes a mixed-type solution for system (14), which represents a superposition of two solitary waves correponding to the acoustical and optical branches at the decoupled case.

Finally we consider the case where $C_1 = C_2 = C$. In such a case we obtain from (22) and (16)

$$\bar{\phi}_j(\zeta) = A_j \operatorname{sech}[C(\zeta - \zeta_0)] \qquad \bar{\psi}_j(\zeta) = B_j \operatorname{sech}[C(\zeta - \zeta_0)] \tag{29}$$

where $\zeta_1 = \zeta_2 = \zeta_0$ and C is given by

$$C^2 = \frac{\delta_1}{2}(A_1^2 + B_1^2) + \frac{\delta_2}{2}(A_2^2 + B_2^2)$$

which corresponds to the case where $\sigma_1 + \sigma_2 = 1$. The above solution is the same as that obtained in [4] by using a direct substitution technique.

4. Conclusions

In this paper, we systematically investigated some analytical properties of the four-wave interaction equations derived previously by the present authors. We showed that the equations have a Hamiltonian structure and there exist at least four conservation laws. Also, the mixed-type solutions of the equations were constructed by using a modified Hirota technique. It is hoped that the results obtained here will help to understand interesting dynamical behaviour of the FWI system through numerical simulations.

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